Technical Notes

Thermal Stresses in a Viscoelastic Cylinder with Temperature Dependent Properties

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An appoximate solution to the problem of transient thermal stresses in a "long" viscoelastic hollow cylinder with temperature-dependent properties is presented. The approximation consists in averaging the value of the vioselastic "Poisson's" ratio over the whole reduced time interval.

The fact that stress distributions in solids are, in general, insensitive to Poisson's ratio variations and the narrow limits of variation of $\nu(\xi)$ leads us to believe that the effect of this approximation will be small. It is shown in Fig. 1 that in the case of the infinite viscoelastic slab the exact and approximate solutions are very close.

The solution is immediately applicable to such practical problems as "stress accumulation due to thermal cycling" and "shrinkage stresses" in viscoelastic cylinders. It is hoped that the present method will replace previous unrealistic methods of analysis that were based on the assumption that viscoelastic material properties are independent of temperature.

Constitutive Relations

We begin with a review of the constitutive relations of an isotropic thermorheologically simple viscoelastic solid with elastic dilatational response under small deformations. In view of the assumed isotropy, two relations are sufficient[1] to describe uniquely the material behavior of the solid, i.e.,

$$s_{ij} = \int_{-\infty}^{t} G(\xi - \xi') \frac{\partial e_{ij}}{\partial \tau} d\tau \tag{1}$$

$$\sigma_{kk} = 3K(\epsilon_{KK} - 3\alpha_0\Theta) \tag{2}$$

where

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{3}$$

$$e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij} \tag{4}$$

and

$$\Theta = \int_{T_0}^{T} \alpha(T)dt$$

T_0 being the temperature of the undisturbed state, allowance being made for the variation of the coefficient of expansion with temperature.

The function ξ denotes the reduced time variable and is defined by the relation

$$\xi = \int_0^t a\{T(x_k, t)\}dt \tag{5}$$

where

$$a = e^{\phi\{T(x_k,t)\}} \tag{6}$$

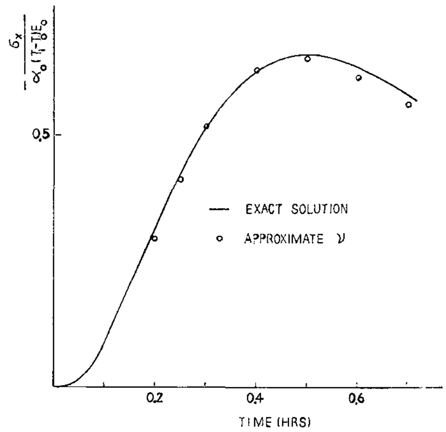

Fig. 1 Stress history at middle plane of infinite slab. Comparison of exact solution[2] against proposed approximate solution.

ϕ being the shift function with respect to the base temperature T_0 that is presumed to be the temperature of the unstressed state

$$\xi' = \xi(x_K, \tau) \tag{7}$$

As a consequence of the shift hypothesis, K is a constant.

For the purpose of our solution, it is convenient to introduce the alternative relation

$$\int_{-\infty}^{t} E(\xi - \xi')\frac{\partial \epsilon_{ij}}{\partial \tau} d\tau = \sigma_{ij} + \int_{-\infty}^{t} \nu(\xi - \xi')\frac{\partial}{\partial \tau} \times (\sigma_{ij} - \sigma_{kk}\delta_{ij})d\tau + \alpha_0\delta_{ij}\int_{-\infty}^{t} E(\xi - \xi')\frac{\partial \Theta}{\partial \tau} d\tau \tag{8}$$

which may be derived from (1) and (2).

The function $E(t)$ and $\nu(t)$, to which we shall refer as tension modulus and "Poisson's ratio," may be found from a uniaxial tensile relaxation test where $E(t)$ and $-\nu(t)$ are the axial stress and lateral strain, respectively, corresponding to a unit step input of axial strain.

Analysis

In cylindrical, polar coordinates

$$r_1 \leqslant r \leqslant r_2, \qquad -\infty < z < \infty, \qquad 0 \leqslant \theta \leqslant 2\pi$$

let the temperature field have the form,

$$T = T_0$$
$$(r_1 \leqslant r \leqslant r_2,\ -\infty < Z < \infty,\ 0 \leqslant \theta \leqslant 2\pi,\ -\infty < t < 0)$$
$$T = T(r, t) \qquad (0 < t < \infty)$$

The cylinder is stress free at $r = r_1$ and is rigidly bonded at $r = r_2$ to an elastic shell with material properties independent of temperature.

In view of the preceding conditions,

$$u_z = u_\theta = 0 \qquad u_r = u(r, t) \tag{9}$$

$$\epsilon_{r\theta} = \epsilon_{\theta z} = \epsilon_{rz} = \epsilon_z = 0 \tag{10}$$

$$\epsilon_r = \partial u/\partial r \qquad \epsilon_\theta = u/r \tag{11}$$

$$\epsilon = \epsilon_r + \epsilon_\theta + \epsilon_z = (\partial u/\partial r) + (u/r) \tag{12}$$

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As a consequence of (9–12), one has the following compatibility relations:

$$\epsilon = (1/r)(\partial/\partial r)(r^2\epsilon_\theta) \tag{13}$$

$$(\epsilon_r - \epsilon_\theta)/r = \partial\epsilon_\theta/\partial r \tag{14}$$

whereas the only equilibrium condition not satisfied identically is

$$(\partial\sigma_r/\partial r) + [(\sigma_r - \sigma_\theta)/r] = 0 \tag{15}$$

Letting

$$\sigma = \sigma_r + \sigma_\theta + \sigma_z \tag{16}$$

and as a consequence of (8),

$$\sigma = \sigma_r + \sigma_\theta + \int_{0-}^{t} \nu(\xi - \xi')\frac{\partial}{\partial\tau} \times (\sigma_r + \sigma_\theta)d\tau - \alpha_0 \int_{0-}^{t} E(\xi - \xi')\frac{\partial\Theta}{\partial\tau}d\tau \tag{17‡}$$

or,

$$\sigma = \int_{0-}^{t} \{H(t-\tau) + \nu(\xi - \xi')\}\frac{\partial}{\partial\tau} \times (\sigma_r + \sigma_\theta)\,d\tau - \alpha_0 \int_{0-}^{t} E(\xi - \xi')\frac{\partial\Theta}{\partial\tau}d\tau \tag{18}$$

It follows from (1, 2, and 8) that

$$\nu(\xi) = \tfrac{1}{2} - \tfrac{1}{3}[E(\xi)/K] \tag{19}$$

Let E_0 denote the glassy value of the tension modulus and E_R the rubbery (equilibrium) value. Then, for all ξ,

$$E_R \leqslant E(\xi) \leqslant E_0$$

Furthermore, since $E(\xi)$ is a monotonically decreasing function of ξ, it follows from (19) that

$$H(t)(1 + \nu_0) \leqslant H(t) + \nu(\xi) \leqslant H(t)(1 + \nu_R) \tag{20}$$

where ν_0 is the glassy value of the Poisson's ratio and ν_R the rubbery value. The latter is in view of (19) and (19a), smaller or equal to 0.5 (depending on the values of K and E_R), and is greater than ν_0. On the other hand, ν_0 is, in general, greater than 0.3; for instance, in the case of polymethyl methacrylate, it has been observed[2] to have a value of 0.35. These arguments lead to the conclusion that $1 + \nu_0$ and $1 + \nu_R$ are sufficiently, close, and, consequently, $H(t) + \nu(\xi)$ varies between narrow limits. Therefore, with sufficient accuracy one may write

$$H(t) + \nu(\xi) = (1 + \boldsymbol{\nu})\,H(t) \tag{21}$$

where $\boldsymbol{\nu}$ is some constant value of ν between ν_0 and ν_R, say,

$$\nu = \tfrac{1}{2}(\nu_0 + \nu_R) \tag{22}$$

By virtue of (21), Eq. (17) takes the simpler form

$$\sigma = (1 + \boldsymbol{\nu})(\sigma_r + \sigma_\theta) - \alpha_0 \times \int_{0-}^{t} E(\xi - \xi')\frac{\partial\Theta}{d\tau}\partial\tau \tag{23}$$

Making use of (2, 13, 15, and 23), one obtains

$$\frac{1}{r}\frac{\partial}{\partial r}(r^2\epsilon_\theta) = \frac{1+\boldsymbol{\nu}}{3K}\frac{1}{r}\frac{\partial}{\partial r} \times (r^2\sigma_r) - \frac{\alpha_0}{3K}\int_{0-}^{t} E(\xi - \xi')\frac{\partial\Theta}{\partial\tau}d\tau \tag{24}$$

Integrating (24),

$$\epsilon_\theta = \mu\sigma_r - \Psi(r, t) + [(t)/r^2] \tag{25}$$

‡ The lower limit of integration has been replaced by 0 − since, prior to $t = 0$, the cylinder remains undisturbed.

where

$$\mu = 1 + \boldsymbol{\nu}/3K$$

$$\Psi(r, t) = \frac{3\alpha_0}{r^2}\int_{r}^{r_2} \rho\Theta d\rho - \frac{\alpha_0}{3Kr^2} \times \int_{r}^{r_2} \rho\left\{\int_{0-}^{t} E(\xi - \xi')\frac{\partial\Theta}{\partial\tau}d\tau\right\}d\rho \tag{26}$$

and

$$\Psi(r_2, t) = 0 \tag{27}$$

We now differentiate (25) with respect to r; then, after differentiation with respect to time, we multiply by $G(\xi - \xi')$ and integrate between the limits of $0-$ and t. The result is

$$\int_{0-}^{t} G(\xi - \xi')\frac{\partial}{\partial\tau}\left(\frac{\partial\epsilon_\theta}{\partial r}\right)d\tau = \mu\int_{0-}^{t} G(\xi - \xi')\frac{\partial}{\partial\tau} \times \left(\frac{\partial\sigma_r}{\partial r}\right)d\tau - \int_{0-}^{t} G(\xi - \xi')\frac{\partial}{\partial\tau}\left(\frac{\partial\Psi}{\partial r}\right)d\tau - \frac{2}{r^3}\int_{0-}^{t} G(\xi - \xi')\frac{dc}{d\tau}d\tau \tag{28}$$

Also, as a direct consequence of (1),

$$\sigma_r - \sigma_\theta = \int_{0-}^{t} G(\xi - \xi')\frac{\partial}{\partial\tau}(\epsilon_r - \epsilon_\theta)\,d\tau \tag{29}$$

In the light of (14, 15, and 29), Eq. (28) becomes

$$-\frac{\partial\sigma_r}{\partial r} = \mu\int_{0-}^{t} G(\xi - \xi')\frac{\partial}{\partial\tau}\left(\frac{\partial\sigma_r}{\partial r}\right)d\tau - \int_{0-}^{t} G(\xi - \xi')\frac{\partial}{\partial\tau}\left(\frac{\partial\Psi}{\partial r}\right)d\tau - \frac{2}{r^3}\int_{0-}^{t} G(\xi - \xi')\frac{dc}{d\tau}d\tau \tag{30}$$

Consider now the functions $\xi(r, t)$ and $\xi' = \xi(r, \tau)$. For a fixed r, the function ξ is monotonically increasing in t and it can, therefore, be inverted in the form

$$t = g(r, \xi) \tag{31}$$

Then

$$f(r, t) = f\{r, g(r, \xi)\} \equiv f^*(r, \xi) \tag{32}$$

Also, if we introduce the notation

$$\partial\sigma_r/\partial r = Q(r, t) \tag{33}$$

$$\partial\Psi/\partial r = P(r, t) \tag{34}$$

and make use of (33) and (34) in (30), we obtain the relation

$$-Q^*(r, \xi) = \mu\int_{0-}^{\xi} G(\xi - \xi')\frac{\partial Q^*}{d\xi'}\partial\xi' - \int_{0-}^{\xi} G(\xi - \xi')\frac{\partial P^*}{\partial\xi'}d\xi' - \frac{2}{r^3}\int_{0-}^{\xi} G(\xi - \xi')\frac{\partial c^*}{\partial\xi'}d\xi' \tag{35}$$

Taking Laplace transform of (35) with respect to ξ' in the sense that

$$\bar{Q}^* = \int_{0-}^{\infty} \bar{e}^{p\xi}Q^*(\xi)d\xi \tag{36}$$

and making use of the condition that the cylinder remains undisturbed at $\xi = 0-$, one obtains

$$\bar{Q}^* = \bar{R}p\bar{P}^* + (2/\gamma^3)\,\bar{R}p\bar{c}^* \tag{37}$$

where

$$\bar{R} = \bar{G}(p)/[1 + \mu p\bar{G}(p)] \tag{38}$$

Taking inverse Laplace transform of (37), reverting to the (r, t) plane, and using (33) and (34), it immediately follows that

$$\frac{\partial\sigma_r}{\partial r} = \int_{0-}^{t} R(\xi - \xi')\frac{\partial P}{\partial\tau}d\mu + \frac{2}{r^3}R(\xi - \xi')\frac{dc}{d\tau}d\tau \tag{39}$$

To facilitate the steps that follow, it is convenient to introduce the function $\Phi\ (r, t)$ defined by the relation

$$\Phi(r, t) = \int_{0-}^{t} R(\xi - \xi') \frac{\partial P}{\partial \tau} d\tau \qquad (40)$$

In terms of this function, (39) assumes the simpler form

$$\frac{\partial \sigma_r}{\partial r} = \Phi(r, t) + \frac{2}{r^3}\int_{0-}^{t} R(\xi - \xi') \frac{dc}{d\tau} d\tau \qquad (41)$$

Upon integration of (41) and as a consequence of the boundary condition $\sigma_r(r_1, t) = 0$, one obtains an explicit relation for σ_r:

$$\sigma_r = \int_{r_1}^{r} \Phi(p, t)\, d\rho + 2\int_{r_1}^{r} \frac{1}{\rho^3}\int_{0-}^{t} \times R(\xi - \xi') \frac{dc}{d\tau} d\tau\, d\rho \qquad (42)$$

The unknown function $c(t)$ will be determined from the condition of continuity of radial stress and displacement at the cylinder-shell interface, which yields the relation

$$\epsilon_\theta(r_2, t) = (1 + \nu_s)\alpha_s\Theta_s - \sigma_r(r_2, t)\ \frac{r_2}{h}\frac{1 - \nu_s^2}{E_s} \qquad (43)$$

where the suffix s pertains to the shell and Θ_s is the average temperature over the shell thickness. In particular, as a direct consequence of (42),

$$\sigma_r(r_2, t) = \int_{r_1}^{r_2} \Phi d\rho + 2\int_{0-}^{t} F(t, \tau) \frac{dc}{d\tau} d\tau \qquad (44)$$

where, after reversal of the order of integration on the right-hand side of (42),

$$F(t,\tau) = \int_{r_1}^{r_2} \frac{1}{r^3} R(\xi - \xi')dr \qquad (45)$$

Also, from (25),

$$\epsilon_\theta(r_2, t) = \mu\sigma_r(r_2, t) + [(ct)/r_2^2] \qquad (46)$$

From (43, 44, and 46), one obtains the following integral equation in terms of $c(t)$:

$$r_2^2(1 + \nu_s)\alpha_s\Theta_s - r_2^2\ \lambda \int_{r_1}^{r_2} \Phi(r, t)dr = c(t) + r_2^2\ \lambda \int_{0-}^{t} F(t, \tau) \frac{dc}{d\tau} d\tau \qquad (47)$$

where

$$\lambda = \frac{1 + \nu}{3K} + \frac{1 - \nu_s^2}{E_s}\left(\frac{r_2}{h}\right)$$

Equation (47) is, essentially, a Volterra integral equation of the second kind, from which $c(t)$ may be determined. Upper and lower bounds to the solution of this equation with the type of Kernal occurring here have been established.[3] Substitution of $c(t)$ in (42) solves the problem completely. The hoop stress may be found immediately from the relation

$$\sigma_\theta = \sigma_r + r\ (r\sigma_r/\partial r) \qquad (48)$$

Also, other boundary conditions such as $\epsilon_\theta(r_2, t) = 0$ (rigid shell) or $\sigma_r(r_2, t) = 0$ (free surface) are particular cases of the situation dealt with previously.

References

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Sudden Expansion of a Bounded Jet at a High Pressure Ratio

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THE free jet from a convergent nozzle at a high pressure ratio expands through a series of expansion waves and is then theoretically refocused to a second node as the expansion waves are reflected as compression waves from a constant pressure boundary. The characteristics of such an underexpanded jet in free air are treated in detail in Ref. 1. The outstanding characteristics of the free jet are that the jet diameter is larger than the nozzle exit diameter, that there is a supersonic area in the center of the jet at a static pressure less than the jet boundary pressure, and that the jet is periodic. If a duct is placed around the jet, the duct has no influence unless the duct diameter is less than the maximum jet diameter, since the constant pressure boundary is maintained by pressure propagation in the space between the duct wall and the jet boundary. However, if the duct wall intersects the jet boundary, the boundary condition is altered so that the expansion waves emanating from the nozzle exit plane are reflected from the solid boundary as expansion waves, and the flow is thus straightened to flow parallel to the duct walls. This results in a nonuniform supersonic flow field. Viscous effects then tend to dissipate the waves and to establish a uniform flow field.

To ascertain that boundary-layer effects would not predominate and would allow the expected flow field to form, a series of tests was conducted over a limited range of test conditions. High pressure air was passed through a nozzle mounted in a cylindrical duct whose area was approximately four times the nozzle area. The flow characteristics were observed by measuring the static pressure along the centerline of the duct-nozzle combination. For comparison, tests were conducted at the same conditions with a convergent-divergent nozzle in place of the convergent nozzle.

As illustrated in Fig. 1, the centerline pressure immediately behind the exit plane of the convergent nozzle drops to a very low value in the same manner that the centerline pressure drops in a free jet. This is followed by a compression to the same pressure level that is obtained by a convergent-divergent nozzle of the same area ratio. This illustrates that, at several diameters from the exit plane of a sudden expansion, the flow field, pressure, and Mach number would be similar to that obtained by a smooth expansion in a convergent-divergent

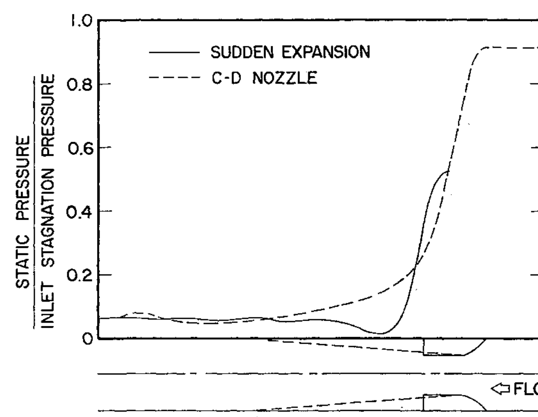

Fig. 1 Centerline pressure of a sudden expansion of a bounded jet.

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